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Some reflexions about complex Poynting's vector

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Résumé.
Cet article présente quelques réflexions sur la signification et la compréhension que l'on peut se faire du vecteur de Poynting quand celui-ci comporte des termes imaginaires. Pour cela on s'appuie sur une représentation d'une ligne sous le formalisme de l'analyse tensorielle des réseaux, puis on transforme la ligne en une succession de cellule réactive se transmettant de proche en proche l'énergie électromagnétique. Ce processus s'apparente à une quantification. Ensuite, en regardant comment le même mécanisme se comporte dans un milieu à pertes on voit le sens du vecteur de Poynting dans de tels milieux.

Abstract
Poynting's vector leads to real value coming from the product of the electric field amplitude with the magnetic field amplitude conjugated. We can assimilate the light propagation with a lossy line. The input of the line behaves like a real load. It translates the dissipation of Poynting's energy in free space. But this lossy line can be itself compared with a sequence of resonators. In each of these resonators, we can see imaginary energies stored. One electrical energy stored in the capacitor and one magnetic energy stored in the self inductance. This kind of approach can be seen as a quantification process. Starting from this fact we can define complex Poynting's vector as a transportation from point to point of imaginary stored energy and no more as a real one. We finally develop this reasoning by trying to make a link between both definition of Poynting's vector.

Mots-clés : Poynting's vector, losses media.

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Introduction

Poynting's vector comes from Maxwell's equation. When we take a look to the demonstration, Poynting's vector explains the energy going out a volume of stored and fluing energy. But it doesn't detail how the energy is distributed before to be received far from the source by another system. A question can be to look inside this transmission of energy to understand what is the nature of the field inside this channel? After that we can hope making some link between the real dissipation of radiated energy that leads to Poynting's vector and the kind of energy inside its transmission.

1) Poynting's vector

Like Jackson, we start from conservation of energy. The product $P = qv.E$ is equivalent of a current per meter multiplied by the electric field $P = idx.E$. As $J.S = i$, this leads to $P = J.Sdx.E$. Finally:

$$P = \int_v dx^3 \mathbf{J} \cdot \mathbf{E} \tag{1}$$

This power must be balanced by the electromagnetic energy stored in the near volume around the currents. We use Maxwell's equations for making appear the field attached with the currents.

$$P = \int_v dx^3 \mathbf{J} \cdot \mathbf{E} = \int_v dx^3 \left[\nabla \times \mathbf{H} - \frac{\mathbf{D}}{dt} \right] \cdot \mathbf{E} \tag{2}$$

Both term inside the bracket can be multiplied by E. Then using the canonical identity :

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$$

with $(\nabla \times \mathbf{E}) = -\partial_t \mathbf{B}$ we obtain :

$$P = \int_v dx^3 \mathbf{J} \cdot \mathbf{E} = -\int_v dx^3 \left[\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\mathbf{B}}{dt} \right] \tag{3}$$

At this step writing the stored electromagnetic energy $u = 1/2 \, (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$ and $\mathbf{S} = (\mathbf{E} \times \mathbf{H})$ we can obtain the equation:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E} = 0 \tag{4}$$

This equation gives all the budget in electromagnetic energy flux. We must find all the forms of this energy: dissipation energy, stored energy and radiated one. We easily find:

1. the stored energy flux associated with $\partial_t u$;

2. the dissipated power associated with $\mathbf{J} \cdot \mathbf{E}$;
3. the radiated power associated with $\nabla \cdot \mathbf{S}$

   $\mathbf{S}$ is Poynting's vector.

2) Lagrangian associated with the budget equation

Seeing equation 4 we can interpret it as the total power involved in a closed circulation where the total potential energy is zero. So we can associated the equation with various laws attached with branches in a graph. In this cellular topological approach, the cinetic energy is associated with the mesh described by the close circuit, like the external sources. The electric stored energy can be associated with a branch like the energy of dissipation. A port presents a real impedance similar to a resistance and a dissipation, but represents the radiated and lost energy. Figure 1 represents the graph we describe just before.

If J is the current that runs into the loop, the electric potential developed accross the capacitive branch C (associated with the electric stored energy) is given by:

$$u_C = \frac{1}{C} \int_t dt J(t) \tag{5}$$

while the voltage developed accross the resistance is $u_R = R.J(t)$.

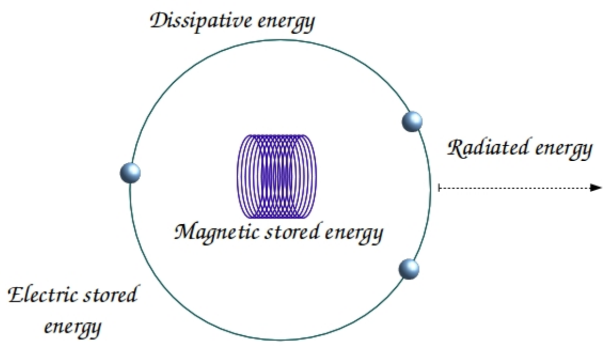

Figure 1: Closed circuit for lagrangian

If ζ is the impedance presented by the radiated energy between its two ports, the potential developed is equal to $u_P = \zeta.J$. And finally, adding the magnetic stored energy and its potential u_L we find the equation associated with the loop:

$$u_C + u_R + u_P + u_L = \frac{1}{C} \int_t dt J(t) + R.J(t) + \zeta.J(t) + L \frac{dJ(t)}{dt} = 0 \tag{6}$$

This is Lagrange's equation associated with the circuit representation of the summation of energies of dissipation, electric stored energy, magnetic stored energy and radiated energy.

We want now to detail what is the significance of z.

3) The input impedance of the radiated energy

The radiated energy can be seen as an infinite guided waves, as no reflected energy can be measured in the input ports of the radiated energy.

This reflexion was already conduct by Feynman. We imagine an infinite succession of LC cells (L being an inductance and C a capacitance). Let's call the serie impedance Z_1 and the paralel impedance Z_2. If Z_0 is the impedance presented by all the network of successive LC cells at its input Z_i, adding Z_0 in parallel to Z_2 gives:

$$Z_i = Z_1 + \frac{Z_2 Z_0}{Z_2 + Z_0}$$

But as Z_0 results from an infinite succession of cells Z_1 in serie with Z_2, to add one cell musn't change the input impedance. And so $Z_i = Z_0$. By replacing Z_i by Z_0 we obtain the expression of Z_0:

$$Z_0 = \frac{Z_1}{2} + \sqrt{\left(\frac{Z_1^2}{4}\right) + Z_1 Z_2} \tag{7}$$

The equation 1 gives the input impedance of an infinite network of successive LC cells. If we go back to the case where $Z_1 = Lp$ and $Z_2 = 1/Cp$ (p being Laplace's operator), we obtain:

$$Z_0 = \sqrt{\left(\frac{L}{C}\right) - \left(\frac{\omega^2 L^2}{4}\right)} \tag{8}$$

If $\omega < 2/\sqrt{LC}$ the input impedance behaves like a real resistance. But the inductance and capacitance values for one cell are determined by:

$$L = \tfrac{Z_0}{v} dx \quad C = \tfrac{1}{Z_0 v} dx$$

and so the pulsation w tends to infinite. It means that a lossless line transports all frequencies without attenuation.

The fact that the line presents a resistance for input impedance was also discussed by Feynman in his course. The transmitted energy to the first cell, even if this cell is purely imaginary is real in a first time. This is the energy necessary to charge the imaginary impedances. Then this energy is transmitted to the next cell, etc. As the processus continues until infinite, no cells remains in a stored energy state and ithis appears like a dissipative process.

The input impedance is called the characteristic impedance, given by:

$$Z_0 = \sqrt{\frac{L}{C}}$$

If we consider a line made of two parallel plates separated by a distance y and of width W. The inductance and the capacitance can be approximated for one cell by:

$$L = \mu_0 \tfrac{ydx}{W} \quad C = \epsilon_0 \tfrac{dxW}{y}$$

so that:

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu_0}{\epsilon_0}\frac{y}{W}} = \eta_0 f$$

η_0 being a wave impedance and f a geometrical function. But we have also:

$$\frac{E}{B} = v_0 \Rightarrow \frac{E}{H} = \mu_0 v_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \eta_0$$

v_0 being the light speed in free space $(\mu_0\epsilon_0)^{-1}$ and the electric and magnetic field following a trans-electromagnetic field like in a lossless line or in free space. Note that a lossless line has zero resistance. It implies that in fact in this imaginary line there is no metallic structures. Any metallic structure implies losses. Finally this line can be seen as a particular mode of propagation where the field remains in a guided finited location with no angular dispersion. But seen from its input, the circuit doesn't know how the field diverge after because of our assumption of infinite propagation with no reflexions. It doesn't change nothing in our minding to consider this lossless line or a perfectly matched antenna, whatever the radiation diagram.

We want now to find the relation between Poynting's vector and the power dissipated in radiation in our circuit. The instantaneous power dissipated on the radiation branch is $P_r = Z_0 J^2(t)$. We can develop this expression:

$$P_r = \eta_0 f J^2 = \frac{E}{H} f J^2 \tag{9}$$

But in general we can write $J = f_B H$ where f_B is a geometrical function. With $f = f_E/f_B$ we obtain:

$$P_r = \eta_0 f J^2 = \frac{E}{H}\frac{f_E}{f_B}\left(H f_B\right)^2 = EH f_E f_B \tag{10}$$

Now Poynting's vector by itself has no real meaning. What is really possible to be felt and measured is its integral on the surface of radiation A:

$$P_r = \int_A d\mathbf{A}\cdot\mathbf{S} \tag{11}$$

Which can be written:

$$P_r = EH \int_A d\mathbf{A} \cdot d\mathbf{f}_E \times d\mathbf{f}_B \tag{12}$$

In the case of guided waves, this leads to equation 10. And so, the power dissipated into the line is a good representation of the Poynting's term in equation 4.

We can finally redraw figure 1, figure 2.

Next step is to look at the line with some different point of view. Each cell can be considered separately like a local field process with coupling functions with the neighbouring cells.

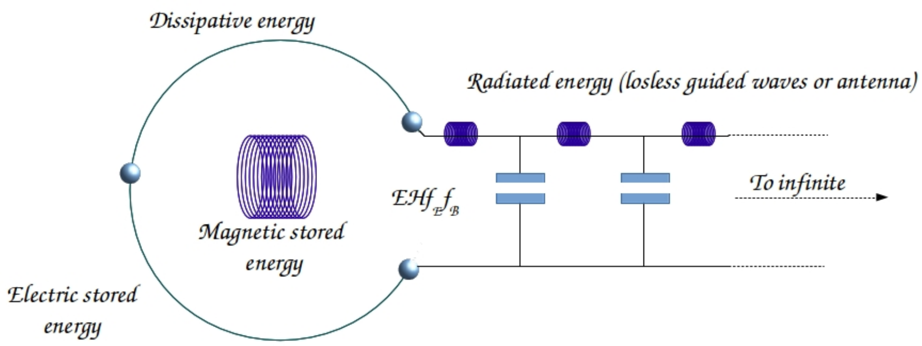

Figure 2: Closed circuit for lagrangian with perfect line

4) Jumping from cells to cells

When we look at the line structure represented figure 2, we recognize a periodic structure where the inductance and the capacitance is always the same until the infinite. Note that we don't care of the length associated with each cell. Implicitely, it's a infinitely short length. Under the tensorial analysis of networks, the impedance matrix of such a structure in the mesh space is this one (we limit its development to the first three cells):

$$z = \begin{bmatrix} Lp + \frac{1}{Cp} & -\frac{1}{Cp} & 0 \\ -\frac{1}{Cp} & Lp + \frac{2}{Cp} & -\frac{1}{Cp} \\ 0 & -\frac{1}{Cp} & Lp + \frac{2}{Cp} \end{bmatrix} \tag{13}$$

The first cell is always different because it constitutes the frontier with the circuit connected to the line. The first inductance can be neglected compared to the inductance of the circuit. If we associate a mesh to each cell in the line structure, next figure 3 leads to the same impedance matrix for the guided waves.

The difference here is that the line is made of separate cells, coupled through the function $-1/Cp$, each cell being a purely LC one. And we know that such a LC circuit represents a resonator. Its working is influenced by the coupling to the neighbouring cells. But before the coupling acts, it supposes that we were able to identify a part of the radiated process like a local resonator. In accordance with the uncertainty principle, we cannot say where this resonator is exactly. If we can locate it with accuracy, it means that we don't know its resonance frequency. But in our abstract graph we can rigorously establish this equivalence without saying where the cell is located, and understand more clearly how the field goes from cell to cell.

We can now study the field characteristics inside this cell.

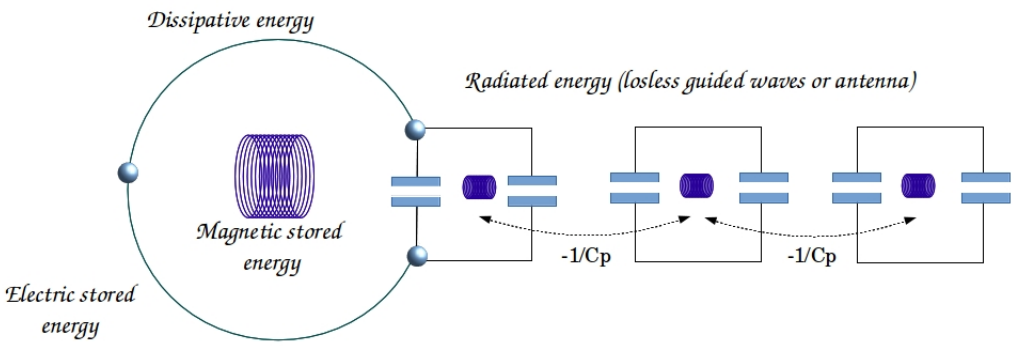

Figure 3: Closed circuit for lagrangian with perfect line with coupled cells

5) Field inside one cell

As one cell behaves like a resonator, it supposes that we were able to separate the field energy in a litlle box of one half wavelength dimension. Inside this box, we can measure a stored electric and magnetic fields energy. When looking to equation 3, it means that this portion of line doesn't represents the radiation any more, but here a local stored energy. The radiation is reported by the coupling with all the other cells following the one we study. Let's take a look to this process. The electromotice force e_c coming from the previous cell is:

$$e_c = -\frac{1}{Cp}\frac{e_0}{Lp + \frac{2}{Cp}} = -\frac{e_0}{2 + LCp^2}$$

but $p = j\omega$ and $\omega = \sqrt{LC}^{-1}$, and so:

$$e_c = -e_0$$

The impulse of the field is transmitted from cell to cell.

Now if we compute the power associated with the cell loaded by the characteristic impedance Z_c that represents all the following cells, we obtain, with the impedance matrix:

$$\zeta = \begin{bmatrix} \frac{2}{Cp} + Lp & -\frac{1}{Cp} \\ -\frac{1}{Cp} & \frac{1}{Cp} + Z_c \end{bmatrix} \tag{14}$$

the total power (including all kind of energies) is:

$$P = \zeta_{xy} J^x J^y = \left(\frac{2}{Cp} + Lp\right)(J^1)^2 - \frac{2}{Cp} J^1 J^2 + \left(Z_c + \frac{1}{Cp}\right)(J^2)^2 \tag{15}$$

But like all the cells are identical, the current from cell to cell should be the same. This imply that $J^1 = J^2 = J$ and by replacement:

$$P = LpJ^2 + \frac{1}{Cp} J^2 + Z_c J^2 \tag{16}$$

We find a similar process as before, i.e. that Poynting's vector appears like a port of dissipation through radiation connected to a resonator circuit. Poynting's vector remain real in that case where the propagation medium is lossless.

6) Complex Poynting's vector

Starting from our succession of lossless cells we can introduce some losses with the inductance. This translates the propagation in a lossy medium. It modifies the impedance matrix by adding a resistance r to the cell mesh and by modifying the characteristic impedance function. It becomes:

$$Z_c = \sqrt{\frac{r + Lp}{Cp}}$$

and the budget 15 is now:

$$P = \left(Lp + \frac{2}{Cp} + r\right) J^2 + \sqrt{\frac{L}{C} + \frac{r}{Cp}} J^2 \tag{17}$$

Noting $Z_0 = \sqrt{LC^{-1}}$ and $\omega_0 = \sqrt{LC}^{-1}$ the last term becomes:

$$\left[\sqrt{\frac{L}{C} + \frac{r}{Cp}}\right] J^2 = \left[\sqrt{Z_0 \left(1 - j\frac{rC\omega_0^2}{\omega}\right)}\right] J^2 \tag{18}$$

if the losses are sufficiently weak, i.e. $rC\omega_0^2\omega^{-1} \to 0$, this leads to:

$$\left[\sqrt{\frac{L}{C} + \frac{r}{Cp}}\right] J^2 = Z_0 \left(1 - j\frac{1}{2}\frac{rC\omega_0^2}{\omega}\right) J^2 \tag{19}$$

We have seen that this term must conduct to Poynting's flux through equation 12. But here, it includes an imaginary part implying that Poynting's vector must be completed to present this kind of component. Or (because Poynting's expression is a story of convention) by keeping Poynting's vector as known and adding the imaginary part as an added part coming from the losses in the medium of propagation. The added function is not trivial. Like in a lossy line, we understand that the fields are no more perpendicular one to eachother, due to the electric field component created by the potential accross the resistance, in other words, by the losses. This discussion was already made with another more classical approach by Jackson.

7) Some values and conclusion

For dielectrics, the angle of losses is defined by $tan\delta = rC\omega$. Equation 19 can be written:

$$Z_0' = Z_0 \left[1 - j\frac{1}{2}tan\delta \left(\frac{\omega_0}{\omega}\right)^2\right] \tag{20}$$

and for the modulus:

$$|Z_0'| = |Z_0| \left[1 + \frac{1}{4}tan^2\delta \left(\frac{\omega_0}{\omega}\right)^4\right]^{\frac{1}{2}} \tag{21}$$

For example if we consider paper, angle of losses can reach $tan\delta = 2.10^{-3}$ and $\epsilon_r = 6$. This leads to:

$$Z_0' \approx Z_0 \left[1 + 10^{-6} \left(\frac{\omega_0}{\omega}\right)^4\right]^{\frac{1}{2}}$$

The ratio ω_0/ω can be equal or inferior to 1. So that the amplitude of Poynting's vector is here modified by a factor $(1 + 10^{-6})^{\frac{1}{2}}$. For glasses, the factor can reach $1,005$. But we must be careful with these values, given for not well controled frequency gap. Anyway these values show that Poynting's complex vector concerns before all metallic media.

As this kind of medium for the electromagnetic propagation is very particular, not to say paradoxical, we understand that Poynting's vector is real in most cases.