ISBN 2-10-000151-5

Revue Internationale de

systēmique

Vol. 5, N° 5, 1991

afcet

DUNOD

AFSCET

Revue Internationale de systémique

Revue Internationale de Sytémique

volume 05, numéro 5, page 493 - 515, 1991

Qualitative analysis of system dynamics models

Javier Aracil, Miguel Toro

REVUE INTERNATIONALE DE SYSTÉMIQUE
Vol. 5, N° 5, 1991, pp. 493 à 515

QUALITATIVE ANALYSIS OF SYSTEM DYNAMICS MODELS

J. ARACIL, M. TORO

Escuela Superior de Ingenieros Industriales [1]

Abstract

System Dynamics supplies conceptual tools to build computer simulation models. These models, as mathematical objects, are nonlinear dynamical systems which can show different long term behavior modes (attractors). The qualitative theory of dynamical systems supplies a body of techniques that allows to classify the attractors and then to elucidate the long-term behaviors of a dynamical system. In this paper steady states and transients are analyzed with qualitative techniques. They are applied to some representative system dynamics models.

Résumé

La Dynamique des Systèmes fournit des instruments théoriques pour la construction des modèles de simulation par ordinateur. Ces modèles sont des systèmes dynamiques non-linéaires qui peuvent exhiber des comportements différents à long terme. La théorie qualitative des systèmes dynamiques fournit un ensemble de techniques qui permet de classer les attracteurs, et d'élucider les comportements à long terme du système dynamique. Dans cet article on analyse par des méthodes qualitatives aussi bien les états stationnaires que les états transitoires. Ils sont appliqués à certains modèles de Dynamique des Systèmes.

1. Introduction

System dynamics was conceived originally by Jay W. Forrester as a system methodology leading to computer simulation. It supplies very interesting tools for building models to simulate the behavior of concrete systems in a wide variety of fields (corporations, urban areas, economic an ecological systems, world models,...). Its main strength rises from the possibility of

1. Universidad de Sevilla, Reina Mercedes s/n, 41012 Sevilla (Spain).

Rev. intern. systémique. 0980-1472 Vol. 5/91/05/ 493 /23/$ 4.30/© Afcet Gauthier-Villars

translating the mental models of the expert on a given concrete system into a dynamical system (set of first order differential equations) that once programmed on a computer simulate the actual behavior of the system. The link between the model structure and the behavior this structure gives rise, allows an analysis of the concrete system which can be used for defining policies to improve the system behavior.

It has been claimed that qualitative data are the basis of system dynamics models. However, the word qualitative has been used by many people with different meanings. For the system dynamicist there is a certain tendency to use this word with a meaning which approximates *linguistic* or non-quantitative (Wolstenholme, 1985). Here, however, we shall use it whith the precise meaning given to it in the *qualitative theory of dynamical systems* (Abraham and Shaw, 1987; Guckenheimer and Holmes, 1983; Hirsch and Smale, 1974). This meaning has a *geometrical* or *topological* connotation. It is our contention that this use is extremely relevant to system dynamics.

It is well known that system dynamics is the study of how *structure* determines *behavior* (*fig.* 1). In this context structure means a network of influences between the elements or parts of a system (Bunge, 1979). The structure gives the system its entity as something different from its parts.

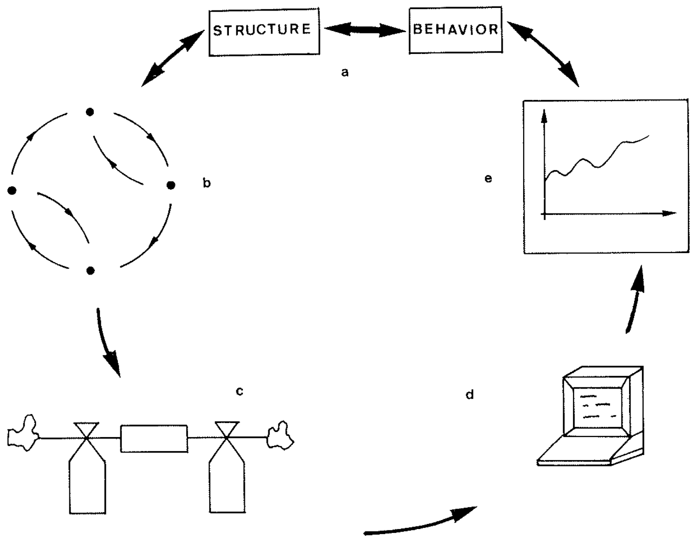

Figure 1. – *Structure and behavior in system dynamics.*

The graphical image of a structure is a graph where the arches are the relations of influence. In system dynamics this graph is called the *causal* or *influences diagram* (*fig.* 1 *b*). It represents an scheme of the mental model of the experts. Behavior to the system dynamicist means *time evolution* of the system variables (*fig.* 1 *e*).

In system dynamics the link between structure and behavior is obtained by a *model* (*fig.* 1 *c*), which is the main conceptual tool. This methodology supplies the means to translate the mental model into the causal diagram (the structure of influences of the system), and then into a dynamical system (a set of first order differential equations) which is usually known as the model of the *concrete system* to be studied. System dynamics is an applied methodology and deals always with concrete systems.

The model, once programmed on a computer (*fig.* 1 *d*), allows the time plots of the system variables to be obtained (*fig.* 1 *e*); and thus its behavior. In this way the link *structure* ⇔ *behavior* is obtained (*fig.* 1 *a*). Looking at this simple chain it would appear that given a structure we get a single behavior mode. The normal use of the computer in simulation strengthens this first impression. However things are not so simple. A given dynamical system can show many behavior modes. One of the objectives of the qualitative analysis is just to elucidate those different modes of behavior.

A model in system dynamics is a mathematical object known as a *dynamical system*. These dynamical systems exhibit a transient behavior after which the evolution of the model tends to reach a permanent regime, which is the long time behavior of the system. In the state space, these long term behavior modes are represented by geometrical objects called *attractors*.

The qualitative analysis of a dynamical system begins with a search of all the attractors the system has. There are three main types of attractors:

1. The *point* or *static attractor*, which corresponds to a stable equilibrium. The behavior of the system is characterized by trajectories that approach the attractor and remain static once reached (*fig.* 2 *a*).

2. The *periodic attractor*, which shows a stable fundamental oscillation. A system approaching the attractor will behave progressively like a perfect oscillation as the time goes on (*fig.* 2 *b*).

3. The *chaotic attractor*, also called *strange attractor*, where the motion is not periodic and does not show any regular pattern (*fig.* 2 *c*).

A given dynamical system can have many attractors. In nonlinear dynamical systems *multiple attractors* are common. Point attractors coexist with periodic and chaotic attractors. This fact is related to the nonlinearities of the system (linear systems show only one attractor, except in degenerate

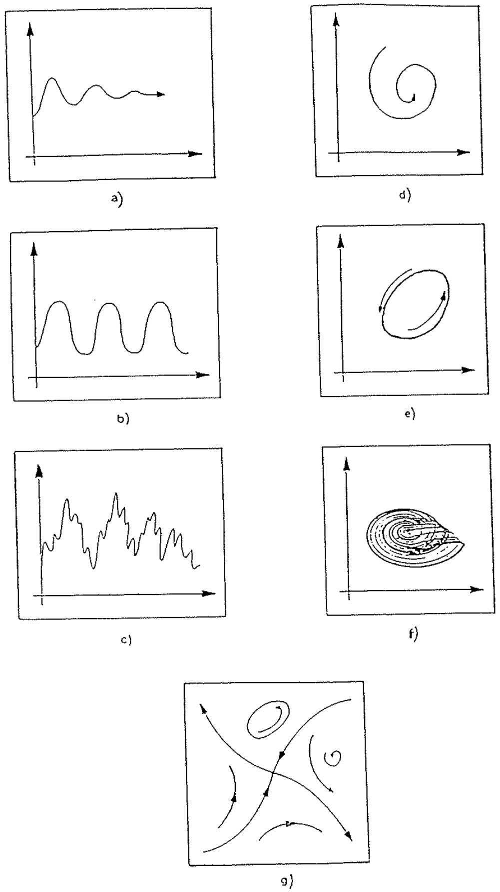

Figure 2. – The three main types of attractors.

cases). We can associate at least one *behavior mode* to every attractor. In this way, by classifying attractors we can have a first insight into the behavior modes a given system can show.

Associated with every attractor is the ensemble of all starting points (initial conditions) that lead to it. This set is the *basin* or *region of attraction* of the given attractor. The basins are enclosed by *separatrices*. The state space of a dynamical system is partitioned in the attraction basins of all its attractors.

The state-space with the attractors, basins and separatrices drawn upon it is called the *phase-portrait* of the dynamical system. The phase-portrait gives a global insight into the behavior modes of the system.

Changes in parameters of a dynamical system determine changes in their phase-portrait. If these changes do not modify the nature of the phase-portrait (the number and type of the attractors), then the system is said to be *structurally stable*. This property ensures the robustness of the model, and thus the conclusions obtained from it; otherwise, a *bifurcation* occurs giving rise to a qualitatively different phase-portrait. *Catastrophe theory* (Poston and Steward, 1978; Thom, 1977; Zeeman, 1977) has provided excelent examples of bifurcations. The main tool to study bifurcations in a dynamical system is the *bifurcation diagram*. These diagrams allow to analyse how changes in the parameters can qualitatively modify the behavior modes of a system.

With the help of qualitative analysis we can get a global perspective on the behavior modes a system can show (phase-portrait), and on how changes in parameters modify these behavior modes (bifurcation diagram). This last can be considered as a sort of *generalized sensitivity analysis*. The interest of sensitivity analysis to system dynamics has long been emphasized (Forrester, 1969, pp. 110-111). The object of this analysis is to study how changes in parameters determine changes in the behavior modes of the system (Ford *et al.* 1985). However, qualitative analysis takes priority to classical sensitivity analysis as it gives a global perspective of the behavior modes, whereas sensitivity analysis only gives a local view (around the nominal trajectory).

The qualitative analysis is based on tools of geometrical nature. These tools take advantage of the graphical possibilities of the computers. They help to develop a deep intuitive comprehension of the mechanism underlying the behavior of the model, and, in this way, to understand how the behavior is generated and how to act in order to modify it. The examples in the following sections will illustrate these possibilities.

The actual implementation of qualitative analysis techniques is practically restricted to small size models. However that is not a great disadvantage as in recent times greater emphasis has been placed on small models in the system dynamics community (Forrester, 1986; Morecroft, 1986). These small models show a greater transparency for the dialogue between "experts mental model" and "simulation model behavior". This dialogue can be greatly extented with the tools supplied by the qualitative analysis techniques.

2. Qualitative analysis of urban dynamics models

Let us now be more concrete and consider an elementary system dynamics model such as the BSNSS3 taken from (Alfred and Graham, 1976). This is one of the simplest models we can find in system dynamics. However its structure is not restricted to this case, and has some generality because it models the *logistic growth*, which is wide-spread. Model BSNSS3 represent the evolution of economic activity in an urban area. Its Forrester diagram is shown in figure 3 *a*. It has a single level variable: the *business structures BS*, which measures the economic activity of the area. It has two rate variables

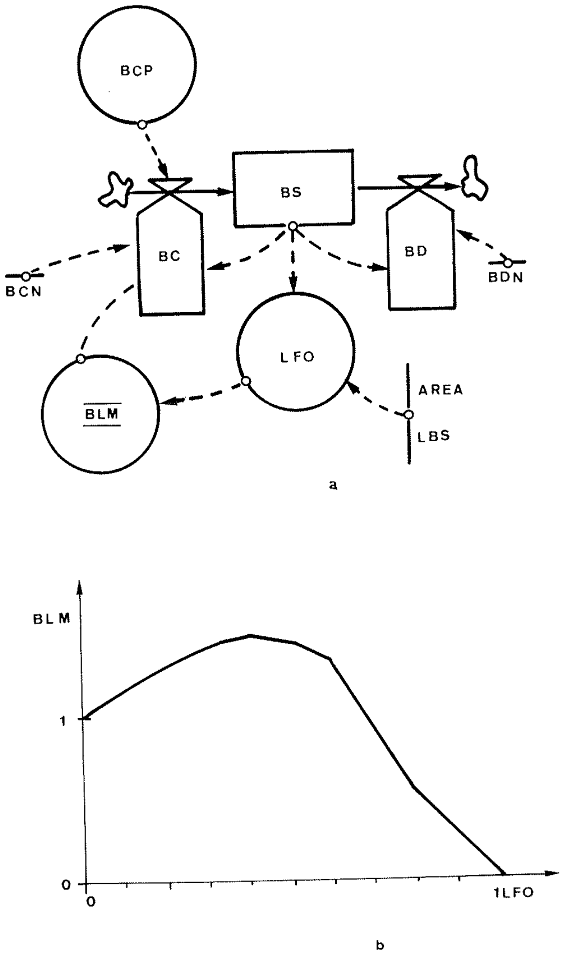

Figure 3. – Diagram and tables of the BSNSS3 model.

(*business construction BC* and *business demolition BD*) each one with an associated parameter (*business construction normal BCN* and *business demolition normal BDN*). The ratio between these two parameters defines parameter $p = BDN/BCN$ which will be used below.

This model can be expressed mathematically in the form of the equation:

$$\frac{d\mathrm{BS}}{dt} = \mathrm{BS}\,(\mathrm{BCN} \times f\,(k \times \mathrm{BS}) - \mathrm{BDN}) \qquad (1)$$

which can be written in a more conventional mathematical form:

$$\dot{y} = y\,(nf\,(ky) - m) \qquad (2)$$

where the dot stands for the derivative with respect to time, $y = \mathrm{BS}$, $n = \mathrm{BCN}$, $m = \mathrm{BDN}$, k is a constant and table function f is defined in figure 3 *b*. For details see the above reference. The table functions will be represented by f. This formalism is most convenient for mathematical manipulations. It is assumed that computer simulation is done with the help of DYNAMO.

Depending on the values of the parameters BCN and BDN, and on the initial conditions, model BSNSS3 shows to different qualitative long-term behavior modes: growth or decline.

One of the main tools for the qualitative analysis of a dynamical system is the *equilibria locus* which is sometimes (although improperly) called the *bifurcation diagram*. It plots the equilibria versus parameters (*fig.* 4). The

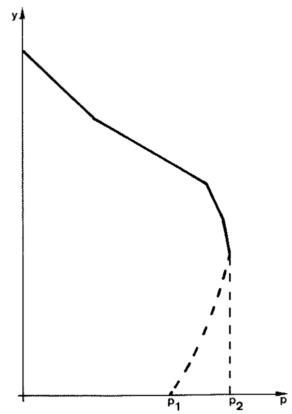

Figure 4. – Equilibria locus of the BSNSS3 model.

equilibria of (1) meet condition $\dot{y}=0$. Hence either

$$y=0 \tag{3}$$

or

$$f(ky)=p \tag{4}$$

where $p=m/n$. The set of points in the plan $(p.y)$ which verify at least one of the Eqs. (3 or 4) is shown in figure 4 and it represents the equilibria locus of (2). To complete the equilibria locus the stability of equilibria must be studied.

Figure 4 is obtained by drawing the stable equilibria with a continuous line, and the unstable ones with a broken line. This equilibria locus can be easily obtained (Aracil, 1981). Computer methods for obtaining the equilibria locus are allowable, based mainly on continuation methods (Mittelman and Weber, 1980; Kubiceck *et al.*, 1983).

An amazing amount of information is compressed in figure 4. This nformation can be displayed in the set of graphs of figures 5. For values of $p<p_1$ the system shows a single attractor. This means that whatever the initial conditions are, the trajectories will have the shape shown in figure 5 *a*. This

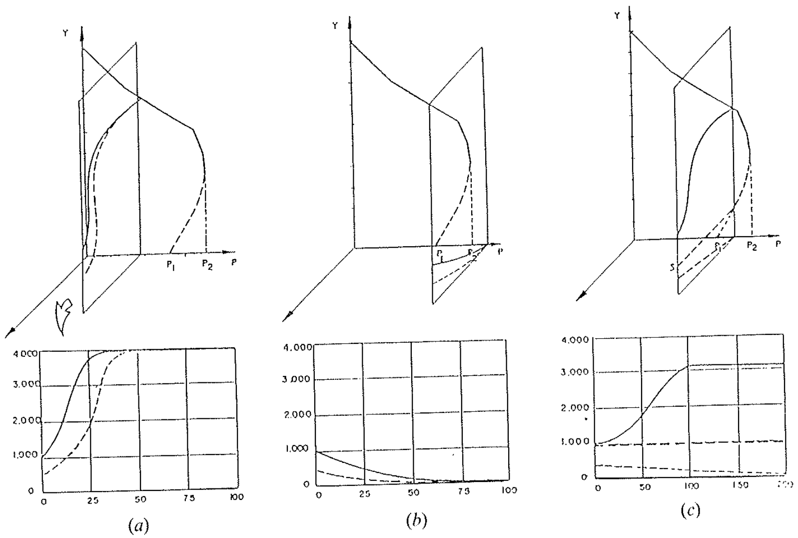

Figure 5. – *Different behavior modes in the BSNSS3 model.*

is the kind of behavior which can be called growth. For $p>p_2$ there is also one attractor, but in this case it represents decline. The number of business tends towards zero for all initial conditions (*fig.* 5 *b*). However, for $p_1<p<p_2$ growth or decline can occur depending on initial conditions (*fig.* 5 *c*). If they are below S in figure 5 *c*, then decline occurs. If they are above S, then growth happens. In this way figure 4 gives a global insight of the different behavior modes the model can show, and can be used as a guide for simulation and for policy implications studies. It should be noted that qualitative has a precise meaning here. We have two qualitatively different behavior modes, associated with two attractors of the dynamical system.

The question that first arises about the viability of the qualitative analysis program for system dynamics is what happens if the model has (as usual) some degree of complexity (the dimension of the model is at least medium sized). Some answers can be given to this problem (Aracil, 1981 *b*, 1984).

Consider the set of models of the Alfeld and Graham book. This is a set of models of growing complexity that can be organized in a hierarchical

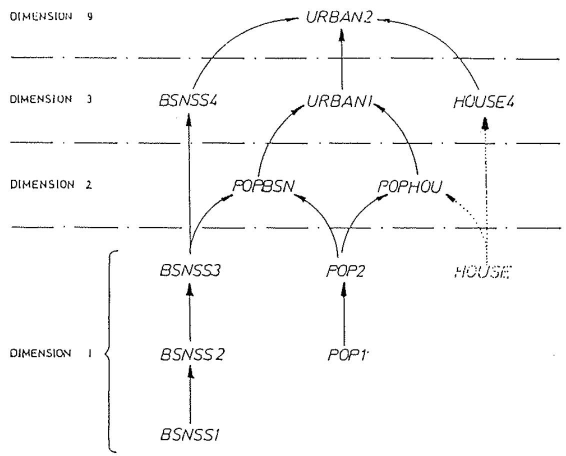

Figure 6. – *Set of urban models in Alfeld and Graham book.*

structure as shown in figure 6. The lowest strata models have the same degree of complexity as the above model BSNSS3. The highest levels are obtained from the lowest ones through a disaggregation process. Qualitative analysis should be applied throughout this process and there are tools to study the appearance of new behavior modes (new bifurcations) through this disaggregation process (Aracil, 1986). It can be shown that under very general conditions, URBAN2 shows the same behavior modes (growth or decline) and are organized in the same ways as for BSNSS3. The disaggregation process, as developed by Alfeld and Graham, does not add any qualitative innovation (in the precise meaning we give here to qualitative) to the modeling process. The details of this approach can be found elsewhere (Toro and Aracil 1986; Toro, 1987).

If we consider the possibility of a spatial disaggregation things are different, and new behavior modes (the loss of homogeneity) can appear. Consider that an urban area is divided into two sub-areas, each one modeled with a model like (2), and that they are connected by a process of linear diffusion with parameter δ (Aracil *et al.*, 1985). This situation can be described by the following equations:

$$\dot{y} = n(f(k(y_1) - p) y_1 + \delta (y_2 - y_1) \qquad (5)$$
$$\dot{y}_2 = n(f(k(y_2) - p) y_2 + \delta (y_1 - y_2)$$

where y_1 and y_2 stand for the business structures in each sub-area. For more details see (Aracil *et al.*, 1985).

Qualitative analysis has allowed parameter regions to be defined where disaggregation does not have a significative effect, and other regions where disaggregation has a considerable effect on the representative capabilities of the model. It should be remarked that the spatial disaggregation just described allows for a link between Forrester system dynamics and Prigogine *dissipative structures* (Nicolis and Prigogine, 1977). Consider a spatial structure formed by elementary modules connected through a diffusion (transport) mechanism. In such spatial structure Forrester system dynamics can deal with the modeling in a module, whereas dissipative structures copes with the spatial inhomogeneous pattern which appears spontaneously due to the instabilizing effects of diffusion.

3. Time-scale decomposition of system dynamics models

To implement the qualitative analysis of some medium-sized dynamical systems the method known as the *two time-scales decomposition* can be applied. It consists of a decomposition of the dynamics of the model into two qualitatively different time-scales: one *fast* and the other *slow*. This allows the study of a dynamical system to be simplified by means of two small dimension subsystems that evolve into two different time-scales. We have some conventional mathematical results that permit us to state that if the equations of a dynamical system are given in the form (Sastry and Desoer, 1981):

$$\frac{dx}{dt} = f(x, y) \qquad (6)$$

$$\varepsilon \frac{dy}{dt} = g(x, y) \qquad (7)$$

where ε is a parameter that takes a small value, then the trajectories of this system starting in (x_0, y_0) can be approximately decomposed into:

(*a*) a *fast motion* in the variable y, with $x = x_0 = \text{Const.}$, and with equation:

$$\frac{dy}{d\tau} = g(x_0, y) \qquad (8)$$

where $\tau = t/\varepsilon$, being τ the fast time-scale.

(*b*) a *slow motion* on the configuration space C, defined by $C = \{(x, y) : g(x, y) = 0\}$, and with equation:

$$\frac{dx}{dt} = f(x, \varphi(x)) \qquad (9)$$

where $g(x, \varphi(x)) = 0$.

It should be noticed that the equilibria of (3) are on C, and consequently the fast motion tries to reach C.

So long as a model of any dimension can be written in the form (6, 7), then the above decomposition can be applied to it, allowing the analysis of the original system to be simplified. In certain cases folds in the configuration space allow problems of qualitative change to be understood, as we shall see below. This decomposition can be applied recursively, so that the original system can be decomposed into two, then into four, and so on. In this way we have, among other things, a method to reduce the dimensionality of a large scale dynamical system.

An example of the applicability of the multiple times-scales analysis is supplied by the Maya civilization collapse model (Hosler *et al.*, 1977). This model tries to simulate the collapse of Maya civilization which took place during the ninth century A. D. This collapse is a great archeological enigma and the model is based on the opinion of expert archeologists, who try to elucidate the mechanism of the collapse. The model is a good example of the interest of system dynamics for Historical Sciences.

After the model was built it was used to try to find policies that could avoid the collapse. This was done by a trial-and-error method and successive simulations of the model. However, even if this latter approach finds some policies that avoid the collapse, a global perspective of how they are generated is lacking. To avoid these shortcomings the method here proposed has been successfully applied and the occurence of the Maya collapse has been geometrically explained through a fold in the configuration space (*fig.* 7).

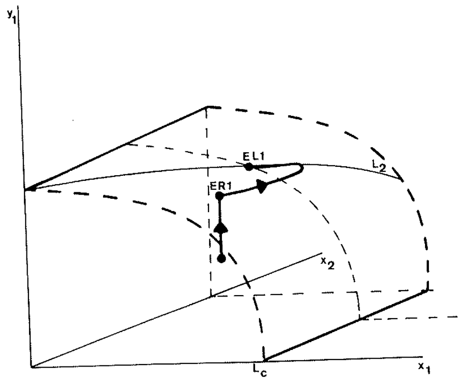

Figure 7. – Configuration space of the MAYA model, with the fold that produces the collapse.

The policies to avoid the collapse are easily defined: they are those that do not reach the fold of the configuration space. In this way we can get a global and geometrical insight into the behavior modes of the system. The policies found by the modellers by simulation are embedded in the wider family that can be found by qualitative analysis. The full details of the analysis can be found elsewhere (Aracil and Toro, 1984).

4. Qualitative analysis of transients in a system dynamics model

In this section the qualitative analysis of a system dynamics World model is considered. It is analyzed a two level model which tries to represent, in a very elementary way, the interactions between *human activity* and the *carrying capacity* of the Earth. This model can be considered a simplified version of the models World-2 (Forrester, 1973) and World-3 (Meadows *et al.*, 1974), and tries to grasp their main qualitative features. It concentrates on the interaction between "human activity" and "carrying capacity" and tries to explain the qualitatively different behavior modes shown in figure 8. The first one represents the collapse of manking, and the second the possibility of a stable accomodation to finite resources.

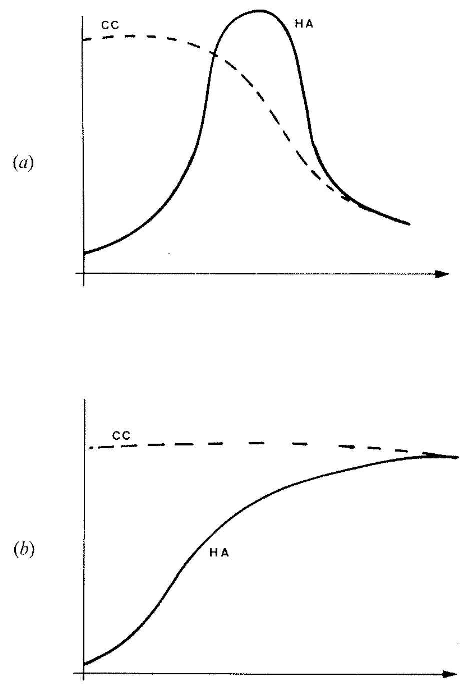

Figure 8. – Behavior modes in the simplified World model: CC carrying capacity; HA human activity.

The diagram of the model is shown in figure 9. The presence of a delay representing the voluntary response to population pressure should be noted. A detailed description of this model can be found in (Randers, 1980, pp. 124-136).

This model is able to show the two main behavior modes that the modelers have tried to represent: 1. the erosion of the carrying capacity, and the subsequent collapse of the human activity (*fig.* 8 *a*); and 2. the material equilibrium obtained through changes in social policies (*fig.* 8 *b*). These two behavior modes were obtained through a parameter adjustment of the model, by trial-and-error and successive runs of the model. With the aid of qualitative analysis we are now better able to understand how these behavior modes are produced an why the transition between both behavior modes happens.

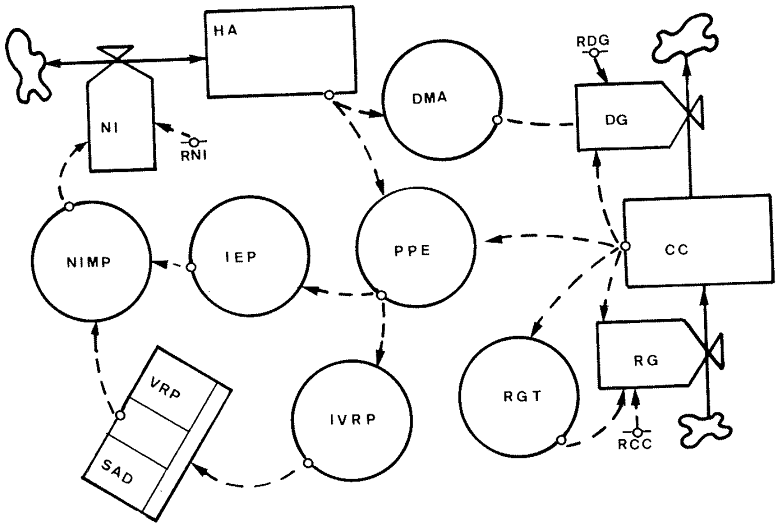

Figure 9. – Diagram of the World model. NI net increase; HA human activity; DMA degeneration multiplier activity; DG degeneration; CC carrying capacity; NIMP net increase multiplier presure; IEP involuntary effect of presure; PPE presure on physical environment; VRP voluntary response to presure; RG regeneration. From Randers, 1980.

To simplify the model a bit further the third order delay has been changed to a first order delay. This does not affect the main qualitative behavior modes of the model. In this way we have a third order dynamical system, whose equations are:

$$\left.\begin{aligned} \frac{dx}{dt} &= nx\,(z+f_1\,(x/y)) \\ \frac{dy}{dt} &= (a-y)/f_3\,(y/a) - by\,f_4\,(x/x_0) \\ \frac{dz}{dt} &= (f_2\,(x/y)-z)/\mathrm{T} \end{aligned}\right\} \qquad (10)$$

where, x stands for the "human activity" HA, y for the "carrying capacity" CC, and z for the "voluntary response to population pressure". Tables are given in figure 10.

If the values of parameter T in this equation take on a large value, then it is clear that the set of equation (10) can be decomposed in two time-scales.

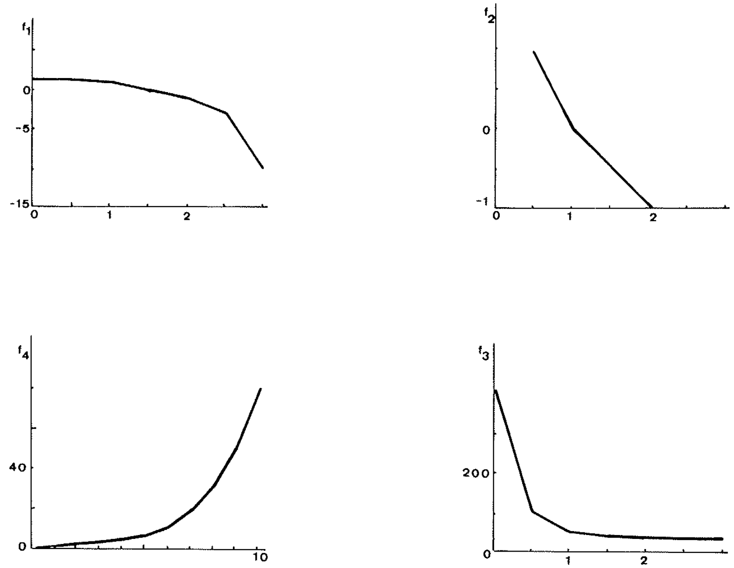

Figure 10. – Tables of the World model.

The third equation represents the slow motion and the two first represent the fast one. The configuration space C is given by the equations:

$$nx\,(z+f_1\,(x/y))=0 \qquad (11)$$

$$(a-y)/f_3\,(y/a)-by\,f_4\,(x/x_0)=0 \qquad (12)$$

According to the conventional results of theory of dynamical systems with two time-scales, the trajectories of the model (10) are formed by a fast motion on the plane $z=z_0=$Const., that tends toward the intersection of this plane with curve C, followed by a slow motion over this curve until it reaches the equilibrium.

When the equilibrium is as in figure 12 trajectories tend to "bend" the hump of figure 11 and it is possible to understand how the hump appears in time trajectories. To avoid this hump, and consequently to avoid the collapse, the equilibrium should be "moved" to the hump of curve C. Furthermore, in this last case a relative optimum of human activity x should be reached. This case is shown in figure 13. This is in fact the solution adopted in (Randers, 1980). However, the analysis developed here supplies a better insight into how it is reached.

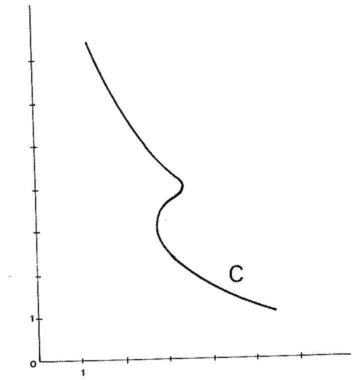

Figure 11. — *Configuration space in the World model and projection of the configuration space on the plane* (x, y).

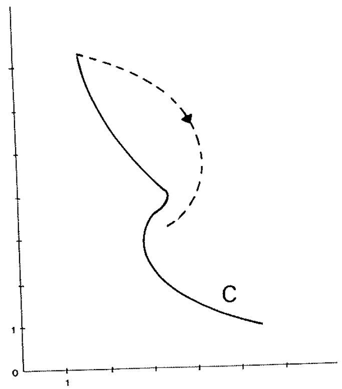

Figure 12. — *Protection of trajectories and the equilibrium on the plane* (x, y) *when the collapse occurs.*

5. Chaos in an autonomous ecological system

In this Section we analyse the chaotic motion of a model which describes the behavior of a prey-predator-food system. Ecological systems supply many interesting examples of concrete systems where modelling techniques can be applied and where interesting cyclic behavior (periodic and aperiodic) occurs. A very interesting and elementary system is the one formed by the food

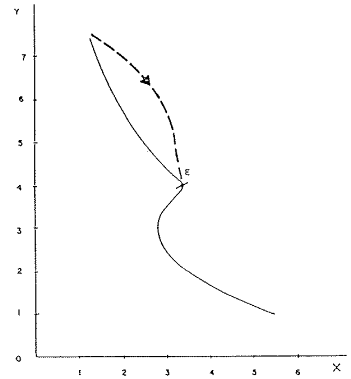

Figure 13. — *Projection of trajectories and the equilibrium on the plane* (x, y) *when the collapse is avoided.*

chain between a predator, a prey and the food for the prey in some habitat with limited resources. For instance, the system formed by deers, wolves and vegetation. Actually these chains show behavior modes where cycles occur, with many different patterns. Of this three-level structure, the sub-structure formed by the predator and the prey, and the one formed by the prey and their food have received some attention in the literature (Taylor, 1984; Peschel and Mende, 1986). However, as far as we know, the full chain has not been studied and it is a very interesting system that can show a huge variety of behaviors.

This system can be be modeled by mixing two well known models: the predator-prey model (Henize, 1971) and the Kaibab plateau model, which copes with the prey-food part of the model (Goodman, 1974). As a matter of fact, the model introduced here can be considered as a modification of the Kaibab type model, where the predators are not extinguished but an interaction between predators and prey of the kind suggested by the Lotka-Volterra models is allowed. This model has previously been introduced in (Toro and Aracil, 1988). The model introduced here is a simplified version of that one, but that retains their main qualitative characteristics. This model shows a great variety of behaviors, including chaos. Previous results can be found in (Toro and Aracil, 1986).

The equations of the model are:

$$\dot{x} = x(g_1(w) - z) \tag{13}$$

$$\dot{z}=b_1 z(x-b_2) \quad (14)$$

$$\dot{y}=g_2(y)-\min(y, x) \quad (15)$$

where x, z and y stand for prey, predator and prey-food respectively and where $w=x/y$. The algebraic form for g_1 and g_2, is given by

$$g_1(w)=-a_1(w-a_2)(w-a_3) \quad (16)$$

$$g_2(y)=-c_1(y-1)(y-c_2) \quad (17)$$

where g_1 represents the prey birth rate, g_2 the regeneration rate and the consumption rate is given by $\min(x, y)$; that is, the minimum between what would consume the prey and the global resources allowable.

It should be observed that for $y=$Const. the two equations (13) and (14) represent a slight modification of the classical Lotka-Volterra predator-prey model. This model is slightly more complex than the Lotka-Volterra one, but has the advantage of being structurally stable. It shows oscillations that are "robust" against perturbations in the model. That model shows point attractors and limit cycles. Between the behaviors associated with both attractors a Hopf bifurcation is produced.

In a similar way, if only equation (14) and (15) are considered one has a model of the interaction of the prey and their food. Then we have a model similar to the Kaibab plateau model (Toro, 1987) which shows a very interesting colapsing behavior, if the resources are exhausted.

For some values of parameters dynamical system (13, 14, 15) shows a chaotic attractor. This type of attractor appears when the value of the parameter b_2 is small. For instance this happens for the set of values of parameters:

$$a_1=0.1, \quad a_2=0.7, \quad a_3=-0.3$$

$$b_1=0.1, \quad b_2=0.05$$

$$c_1=0.5, \quad c_2=-0.1$$

System (13, 14, 15) can be analysed as a two time-scales dynamical system. Then a geometrical understanding of the chaotic attractor can be obtained. The slow time-scale is associated to variables x and z, whereas the fast one is to variable y.

To decompose system (1,2,3) in these two time-scales, first make the transformation $Z=a_1 z$ and for simplicity write z for Z. The new dynamical

system can be written:

$$x=a_1 x(-(w-a_2)(w-a_3)-z) \quad (18)$$

$$z=b_1 z(x-b_2) \quad (19)$$

$$y=g_2(y)-\min(y, x) \quad (20)$$

For small values of a_1 and b_1 (which in our actual case are of the order of 0.1) system (18, 19, 20) has the required canonical form. The configuration space (equilibria set of the fast dynamics) is given by the surface:

$$g_2(y)-\min(y, x)=0 \quad (21)$$

The system motion will be formed by a fast motion towards one of the stable sheets of the configuration surface and a slow motion on that surface (*fig.* 14). The slow motion on the upper sheet ($y>y_m$), where y_m is such that $g_2'(y_m)=0$, has an unstable equilibrium with complex eigenvalues. This gives rise to a helicoidal trajectory which grows until the border of the sheet is reached. Then it falls with a fast motion to the down sheet. The slow motion has no actual equilibrium on the down sheet, but the trajectories tend towards a virtual equilibrium out of this sheet. The full trajectory on the attractor shown in figure 14 is obtained joinning the fast and the show motions. This figure makes clear the reinjection phenomenon, which supplies an intuitive insight of the chaos mechanism.

Figure 15 shows the Poincaré map for this case. The shape of the map is similar to others well known in the literature and suggests the chaotic nature of the attractor. The Lyapunov exponents have been computed to confirm

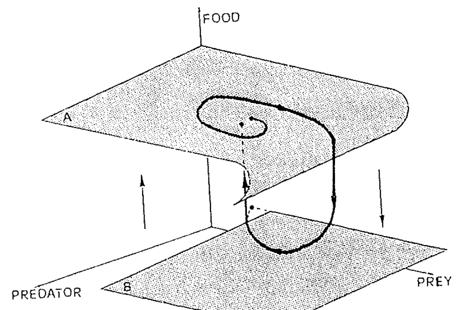

Figure 14. – *Reinjection phenomenon in an ecological system.*

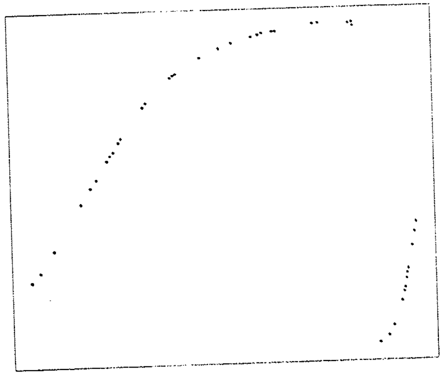

Figure 15. – *Poincaré map for the autonomous ecological system.*

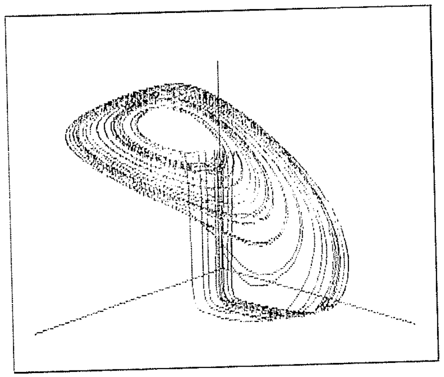

Figure 16. – *Actual trajectory of the attractor of the autonomous ecological system.*

this nature. The values obtained have been:

$$\lambda_1 = 1.29\,E-3 \pm 5.E-5$$
$$\lambda_2 = -1.27\,E-2 \pm 4.E-5$$
$$\lambda_3 = -1.78\,E-1 \pm 3.E-3$$

The fact that $\lambda_1 > 0$ confirms the sensitivity to the initial conditions. That means that two points arbitrarely close to each other will become increasiling

separated for enough long times. This sensitivity is one the most remarkable characteristics of the chaotic behavior. The algorithm used to compute the Liapunov exponents is due to Wolf *et al.* Figure 16 shows a actual trajectory on the attractor, which shows the same form as figure 14.

6. Conclusions

The relevance of qualitative analysis to System Dynamics models has been shown in this paper. The techniques supplied by the qualitative theory of non-linear dynamical system allows us to study systematically the types of attractors that a given system can show, and how they are organized. In this way a global insight into the system behavior modes can be reached, which can be of a great practical interest as a guide to simulation.

The interaction between mental models and all the behavior modes they can generate, which has been emphasized by Forrester (Forrester, 1986), can be better understood with the help of the qualitative techniques. Using a dynamical system only to program a computer and to get trajectories of the models variables evolution is a poor and restrictive use of that model. We can learn many more things about the link between structure and behavior by analyzing that model with the techniques supplied by the modern theory of dynamical non-linear systems. Furthermore, the adoption of these techniques can serve to counteract certain criticism raised against System Dynamics (Berlinski, 1976).

In this paper the long time behavior of urban and ecological models and the transient behavior of an elementary World model has been qualitatively analyzed. An interaction between the qualitative analysis developed here and computer simulation, which allows to obtain the trajectories, is necessary to dig out all the conclusions a System Dynamics model can supply.

References

R. H. ABRAHAM & C. D. SHAW, Dynamics: a visual introduction, in *Self-Organizing System*, F. E. YATES Ed., Plenum Press, 1987.

L. ALFELD & A. GRAHAM, *Introduction to Urban Dynamics*, Wright-Allen Press, 1976.

J. ARACIL, Structural Stability of low-order system dynamics models, *Internat. J. Systems Sci.*, 12, 1981 *a*, pp. 423-441.

J. ARACIL, Further results on structural stability of urban dynamics models, in *System Dynamic and the Analysis of Change*, E. PAULRÉ Ed., North-Holland, 1981 *b*.

J. ARACIL, Qualitative methods in social systems modelling, in *Mathematical Modelling in Science and Technology*, AVULA *et al.* Ed., Pergamon Press, 1983, pp. 125-132.

J. ARACIL, Qualitative analysis and Bifurcations in System Dynamics Models, *IEEE-SMC*-14, 4, 1984, pp. 688-696.

J. ARACIL, Bifurcations and structural stability in the dynamical systems modeling process, *Systems Research*, 3, 4, 1986, pp. 243-252.

J. ARACIL & M. TORO, A case study of qualitative change in system dynamics, *Internat. J. Systems Sci.*, 15, 1984, pp. 575-599.

J. ARACIL, E. PONCE & E. FREIRE, Order through fluctuations and system dynamics models, *Environment and Planning B*, 12, 1985, pp. 103-112.

M. BUNGE, *A World of systems*, Reidel, 1979

D. BERLINSKI, *On systems Analysis*, M.I.T. Press, 1976.

P. CVITANOVIC Ed., *Universality in Chaos*, Adam Hilger Ltd, Bristol, 1984.

A. FORD, J. AMLIN & G. BACKUS, A practical approach to sensitivity testing of system dynamics models, 1983 *System Dynamics Conference*, 1983.

J. W. FORRESTER, *Industrial Dynamics*, M.I.T. Press, 1961.

J. W. FORRESTER, *Urban Dynamics*, M.I.T. Press, 1969.

J. W. FORRESTER, Lessons from system dynamics modeling, 1986 *System Dynamics Conference*, Sevilla, October 22-24, 1986, pp. 1-16.

M. R. GOODMAN, *Study notes in System Dynamics*, Wright-Allen Press, 1974.

J. GUCKENHEIMER & P. HOLMES, *Nonlinear oscillations, dynamical system and bifurcations of vector fields*, Springer-Verlag, 1983.

J. HENIZE, *Course materials for Principles of Dynamic Systems*, M.I.T. Press, 1971.

W. HIRSCH & S. SMALE, *Differential equations, dynamical systems and linear algebra*, Academic Press, 1974.

D. HOSLER, J. A. SABLOFF & D. RUNGE, *Social Process in Maya Prehistory*, Academic Press, 1973.

M. KUBICECK & M. MAREK, *Computational methods in bifurcation theory and dissipative structures*, Springer-Verlag, 1983.

D. L. MEADOWS *et al.*, *Dynamics of growth in finite world*, Wright-Allen Press, 1974; H. D. MITTELMANN & H. WEBER, *Biffurcation problems and their numerical solution*, Birkhauser Verlag, 1980.

E. MOSEKILDE, J. ARACIL & P. ALLEN, Instabilities and chaos in non-linear dynamic systems, *System Dynamics Rev.*, 4, 1988, pp. 14-55.

E. MOSEKILDE *et al.*, Autonomous chaotic behaviour in a generic resource allocation problem, 1986 *System Dynamics Conference*, 1986, pp. 339-351.

T. POSTON & I. N. STEWARD, *Catastrophe Theory and its Applications*, Pitman, 1978.

G. NICOLIS & I. PRIGOGINE, *Self-organization in Non-equilibrium systems*, Wiley, 1977.

J. RANDERS, *The Elements of the System Dynamics Method*, M.I.T. Press, 1980.

E. RASMUSSEN, E. MOSEKILDE & J. D. STERMAN, Bifurcations and chaotic behavior in a simple model of the economic log wave, *System Dynamics Rev.*, 1, 1, 1985, pp. 92-110.

S. S. SASTRY & C. A. DESOER, Jump behavior of circuits and systems, *IEEE-CAS*-28, 1981, pp. 1109-1124.

H. G. SCHUSTER, *Deterministic chaos*, Phisik-Verlag, 1984.

R. THOM, *Stabilité structurelle et morphogenèse*, InterEditions, Paris, 1977.

J. M. T. THOMPSON & H. B. STEWARD, *Nonlinear Dynamics and Chaos*, Wiley, 1986.

M. TORO, Análisis cualitativo y caos en Dinámica de Sistemas, *P. D. Thesis*, Universidad de Sevilla, 1987.

M. TORO & J. ARACIL, Bifurcations and aggregation in Large Scale Systems, *4th IFAC/IFORS Symposium on Large Scale Systems*, Zurich, August 26-29, 1986, pp. 137-140.

M. TORO & J. ARACIL, Qualitative Analysis of System Dynamics Ecological Models, *System Dynamics Rev.*, 4, 1988, pp. 56-80.

E. F. WOLSENHOLME, A methodology for Qualitative system dynamics, 1985 *System Dynamics Conference*, 1985, pp. 1049-1057.

E. C. ZEEMAN, *Catastrophe theory: selected papers*, Addison-Wesley, 1977.

A. WOLF *et al.*, Determining Lyapunov exponents from a time series, *Physica* 16*D*, 1985, pp. 285-317.